

The Terwilliger algebra of the incidence graphs of Johnson geometry

Qian Kong Benjian Lv Kaishun Wang*

Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing, 100875, China

Abstract

Levstein and Maldonado [F. Levstein, C. Maldonado, The Terwilliger algebra of the Johnson schemes, Discrete Mathematics 307 (2007) 1621–1635] computed the Terwilliger algebra of the Johnson scheme $J(n, m)$ when $3m \leq n$. In this paper, we determine the Terwilliger algebra of the incidence graph $J(n, m, m+1)$ of Johnson geometry when $3m \leq n$, give two bases of this algebra, and calculate its dimension.

AMS classification: 05E30

Key words: Terwilliger algebra; incidence graph; Johnson geometry

1 Introduction

Let $\Gamma = (X, R)$ denote a simple connected graph with the vertex set X and the edge set R . For vertices x and y , $\partial(x, y)$ denotes the *distance* between x and y , i.e., the length of a shortest path connecting x and y . Fix a vertex $x \in X$. Let $D(x) := \max\{\partial(x, y) \mid y \in X\}$ denote the *diameter with respect to x* . For each $i \in \{0, 1, \dots, D(x)\}$, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ and define $E_i^* = E_i^*(x)$ to be the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with yy -entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } y \in \Gamma_i(x), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} = \mathcal{T}(x)$ be the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ and $E_0^*, E_1^*, \dots, E_{D(x)}^*$. Then \mathcal{T} is called the *Terwilliger algebra* of Γ with respect to x . Let $V = \mathbb{C}^X$ denote the vector space over the complex number field \mathbb{C} consisting of column vectors whose coordinates are indexed by X . A \mathcal{T} -module is any subspace $W \subseteq V$ such that $\mathcal{T}W \subseteq W$. We call a nonzero \mathcal{T} -module W *irreducible* if it does not properly contain a nonzero \mathcal{T} -module. An irreducible \mathcal{T} -module W is *thin* if $\dim E_i^* W \leq 1$ for every i , and the graph Γ is said to be *thin with respect to x* if every irreducible $\mathcal{T}(x)$ -module is thin.

Terwilliger [12, 13, 14] initiated the study of the Terwilliger algebra of association schemes, which has been used to study group schemes [1, 2], strongly regular graphs [16], bipartite and almost bipartite P - and Q -polynomial association schemes [4, 5], 2-homogeneous bipartite distance-regular graphs [6], the Hypercube [7], the Hamming schemes [9] and the Johnson schemes [10], etc.

*Corresponding author. E-mail address: wangks@bnu.edu.cn

Let Ω be a set of cardinality n and let $\binom{\Omega}{i}$ denote the set of all i -subsets of Ω . The incidence graph $J(n, m, m+1)$ of the Johnson geometry is a bipartite graph with a bipartition $\binom{\Omega}{m} \cup \binom{\Omega}{m+1}$, where $y \in \binom{\Omega}{m}$ and $z \in \binom{\Omega}{m+1}$ are adjacent if and only if $y \subseteq z$. It is known that $J(n, m, m+1)$ is *distance-biregular* (see [3]).

Levstein and Maldonado [10] determined the Terwilliger algebra of the Johnson scheme $J(n, m)$ when $3m \leq n$. Motivated by this result, in this paper we shall determine the Terwilliger algebra of $J(n, m, m+1)$ with respect to $x \in \binom{\Omega}{m}$ when $n \geq 3m$.

This paper is organized as follows. In Section 2, we introduce the intersection matrices and give some useful identities. In Section 3, we determine the Terwilliger algebra of $J(n, m, m+1)$, and show $J(n, m, m+1)$ is thin with respect to x . In Section 4, we give two bases of the Terwilliger algebra and compute its dimension.

2 Intersection matrices

In this section we first introduce the inclusion matrices of a set, then discover the relationship between the adjacency matrix of $J(n, m, m+1)$ and the inclusion matrices, and give some identities for intersection matrices.

The following lemma is useful.

Lemma 2.1 *Let $J(n, m, m+1)$ be the incidence graph of Johnson geometry with a bipartition $\binom{\Omega}{m} \cup \binom{\Omega}{m+1}$. Pick $x \in \binom{\Omega}{m}$. Then $\partial(x, z) = 2i$ if and only if $|z| = m$ and $|x \cap z| = m - i$; $\partial(x, z) = 2i + 1$ if and only if $|z| = m + 1$ and $|x \cap z| = m - i$. Furthermore, when $n \geq 2m + 1$ we have $D(x) = 2m + 1$.*

Proof. Immediate from [8, Lemma 2.2 (1)(3)]. \square

Fix $x \in \binom{\Omega}{m}$. We then consider the adjacency matrix A of $J(n, m, m+1)$ as a block-matrix with respect to the partition $\{x\} \cup \Gamma_1(x) \cup \cdots \cup \Gamma_{2m+1}(x)$. In order to describe the blocks of A , we need to introduce the inclusion matrices.

Let V be a set of cardinality v . The *inclusion matrix* $W_{i,j}(v)$ is a $(0, 1)$ -matrix whose rows and columns are indexed by $\binom{V}{i}$ and $\binom{V}{j}$, respectively, with the yz -entry defined by

$$(W_{i,j}(v))_{yz} = \begin{cases} 1, & \text{if } y \subseteq z, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$W_{i,j}(v)W_{j,k}(v) = \binom{k-i}{j-i} W_{i,k}(v). \quad (1)$$

Let $A_{i,j}$ be the submatrix of A with rows indexed by $\Gamma_i(x)$ and columns indexed by $\Gamma_j(x)$.

Lemma 2.2 *Let $I_{\binom{v}{k}}$ be the identity matrix of size $\binom{v}{k}$. Then*

$$A_{i,j} = 0 \quad (0 \leq i \leq j \leq 2m+1 \text{ and } i \neq j-1), \quad (2)$$

$$A_{2i,2i+1} = I_{\binom{m-i}{m-i}} \otimes W_{i,i+1}(n-m) \quad (0 \leq i \leq m), \quad (3)$$

$$A_{2i+1,2i+2} = (W_{m-i-1,m-i}(m))^t \otimes I_{\binom{n-m}{i+1}} \quad (0 \leq i \leq m-1), \quad (4)$$

where “ \otimes ” denotes the Kronecker product of matrices.

Proof. (2) is directed.

Pick $y \in \Gamma_{2i}(x)$, $z \in \Gamma_{2i+1}(x)$. By Lemma 2.1 we have $|y| = m$, $|z| = m + 1$, $|x \cap y| = |x \cap z| = m - i$. Suppose $y = \alpha_{m-i}\beta_i := \alpha_{m-i} \cup \beta_i$, $z = \alpha'_{m-i}\beta'_{i+1}$, where α_{m-i} and $\alpha'_{m-i} \in \binom{x}{m-i}$, while $\beta_i \in \binom{\Omega \setminus x}{i}$ and $\beta'_{i+1} \in \binom{\Omega \setminus x}{i+1}$. Then

$$(A_{2i,2i+1})_{yz} = (I_{\binom{m}{m-i}} \otimes W_{i,i+1}(n-m))_{yz} = \begin{cases} 1, & \text{if } \alpha_{m-i} = \alpha'_{m-i} \text{ and } \beta_i \subseteq \beta'_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

which leads to (3).

Similarly, (4) holds. \square

Let $C_{i,j}^l(v)$ be a matrix with rows indexed by $\binom{V}{i}$ and columns indexed by $\binom{V}{j}$, whose yz -entry is defined by

$$(C_{i,j}^l(v))_{yz} = \binom{|y \cap z|}{l}.$$

Let $H_{i,j}^l(v)$ be a $(0,1)$ -matrix whose rows and columns are indexed by elements of $\binom{V}{i}$ and $\binom{V}{j}$, respectively, and the yz -entry is defined by

$$(H_{i,j}^l(v))_{yz} = \begin{cases} 1, & \text{if } |y \cap z| = l, \\ 0, & \text{otherwise.} \end{cases}$$

These two matrices may be considered as *intersection matrices* in the sense that the yz -entry only depends on $|y \cap z|$. Observe $C_{i,j}^0(v)$ is the all-one matrix and $C_{i,j}^{\min(i,j)}(v) = W_{i,j}(v)$ ($i \leq j$) or $(W_{j,i}(v))^t$ ($i > j$). We adopt the convention that $C_{i,j}^l(v) = 0$ for any integer l such that $l < 0$ or $l > \min(i, j)$. Note that

$$C_{i,j}^l(v) = \sum_{g=l}^{\min(i,j)} \binom{g}{l} H_{i,j}^g(v). \quad (5)$$

Lemma 2.3 *Let V be a set of size v . Write $W_{i,j} = W_{i,j}(v)$ and $C_{i,j}^l = C_{i,j}^l(v)$. Then*

- (i) $W_{i,j}^t W_{i,k} = C_{j,k}^i$.
- (ii) $C_{i,j}^l W_{j,k} = \binom{k-l}{j-l} C_{i,k}^l$.
- (iii) $W_{i,k} W_{j,k}^t = \sum_{l=\max(0,i+j-k)}^{\min(i,j)} \binom{v-i-j}{k-i-j+l} C_{i,j}^l$.
- (iv) $W_{i,j} C_{j,k}^l = \sum_{h=\max(0,l+j-i)}^{\min(l,i)} \binom{v-l-i}{j-l-i+h} \binom{k-h}{l-h} C_{i,k}^h$.
- (v) $C_{i,j}^l C_{j,k}^s = \sum_{h=\max(0,l+s-j)}^{\min(l,s)} \binom{v-l-s}{j-l-s+h} \binom{i-h}{l-h} \binom{k-h}{s-h} C_{i,k}^h$.

Proof. (i) See [11].

(ii) Immediate from (1) and (i).

(iii) We claim that

$$W_{i,i+1} W_{j,i+1}^t = (v-i-j) C_{i,j}^j + C_{i,j}^{j-1} \quad (j \leq i+1). \quad (6)$$

When $j = i+1$, $W_{i,i+1} W_{j,i+1}^t = W_{i,j} = C_{i,j}^{j-1}$, (6) holds. We now assume $j \leq i$. For any

$y \in \binom{V}{i}$ and $z \in \binom{V}{j}$,

$$\begin{aligned}
& (W_{i,i+1} W_{j,i+1}^t)_{yz} \\
&= \sum_{w \in \binom{V}{i+1}} (W_{i,i+1})_{yw} (W_{j,i+1}^t)_{wz} \\
&= |\{w \mid (y \cup z) \subseteq w, w \in \binom{V}{i+1}\}| \\
&= \begin{cases} v-i, & |y \cap z| = j, \\ 1, & |y \cap z| = j-1, \\ 0, & |y \cap z| \leq j-2, \end{cases}
\end{aligned}$$

which implies (6).

Next we show that

$$W_{i,k} W_{j,k}^t = \sum_{s=0}^{\min(k-i,j)} \binom{v-i-j}{k-i-s} C_{i,j}^{j-s}. \quad (7)$$

Observe (7) holds when $i = k$. By induction, (1), (6) and (i),

$$\begin{aligned}
& W_{i-1,k} W_{j,k}^t \\
&= \frac{1}{k-i+1} W_{i-1,i} W_{i,k} W_{j,k}^t \\
&= \frac{1}{k-i+1} \sum_{s=0}^{\min(k-i,j)} \binom{v-i-j}{k-i-s} W_{i-1,i} C_{i,j}^{j-s} \\
&= \sum_{s=0}^{\min(k-i,j)} \binom{v-i-j}{k-i-s} \left(\frac{v-i-j+s+1}{k-i+1} C_{i-1,j}^{j-s} + \frac{s+1}{k-i+1} C_{i-1,j}^{j-s-1} \right) \\
&= \sum_{s=0}^{\min(k-i+1,j)} \binom{v-i-j+1}{k-i-s+1} C_{i-1,j}^{j-s}.
\end{aligned}$$

Then (7) is obtained by induction, concluding (iii).

(iv) Immediate from (i), (ii) and (iii).

(v) Obtained by (i), (ii) and (iv). \square

3 The Terwilliger algebra

In this section we fix $x \in \binom{\Omega}{m}$, then consider the Terwilliger algebra $\mathcal{T} = \mathcal{T}(x)$ of $J(n, m, m+1)$ when $n \geq 3m$.

For $0 \leq i, j \leq 2m+1$, any matrix M indexed by elements in $\Gamma_i(x) \times \Gamma_j(x)$ can be embedded into $\text{Mat}_X(\mathbb{C})$ by

$$L(M)_{\Gamma_k(x) \times \Gamma_l(x)} = \begin{cases} M, & \text{if } k = i \text{ and } l = j, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \leq i, j \leq 2m+1$, let

$$\begin{aligned}
& \mathcal{M}_{i,j} \\
&= \text{Span}\{C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^l(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^s(n-m), \\
& \quad 0 \leq l \leq \min(m - \lfloor \frac{i}{2} \rfloor, m - \lfloor \frac{j}{2} \rfloor), \quad 0 \leq s \leq \min(\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil)\}.
\end{aligned}$$

Let

$$\mathcal{M} = \bigoplus_{i,j=0}^{2m+1} L(\mathcal{M}_{i,j}), \quad (8)$$

where $L(\mathcal{M}_{i,j}) = \{L(M) \mid M \in \mathcal{M}_{i,j}\}$.

Note that \mathcal{M} is a vector space. By Lemma 2.3 (v) we have \mathcal{M} is an algebra. In the remaining of this section we shall prove $\mathcal{T} = \mathcal{M}$.

We begin with a lemma.

Lemma 3.1 *The Terwilliger algebra \mathcal{T} is a subalgebra of \mathcal{M} .*

Proof. By Lemma 2.2 we have $A \in \mathcal{M}$. For $0 \leq i \leq 2m+1$, since

$$E_i^* = E_i^*(x) = L(C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^{m-\lfloor \frac{i}{2} \rfloor}(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^{\lceil \frac{i}{2} \rceil}(n-m)) \in \mathcal{M},$$

we get $\mathcal{T} \subseteq \mathcal{M}$. \square

For $0 \leq i, j \leq 2m+1$, let $\mathcal{T}_{i,j} = \{M_{i,j} \mid M \in \mathcal{T}\}$, where $M_{i,j}$ is the submatrix of M with rows indexed by $\Gamma_i(x)$ and columns indexed by $\Gamma_j(x)$. Since \mathcal{T} is an algebra, each $\mathcal{T}_{i,j}$ is a linear space. Since $\mathcal{T}E_j^*\mathcal{T} \subseteq \mathcal{T}$, $(\mathcal{T}E_j^*\mathcal{T})_{i,k} \subseteq \mathcal{T}_{i,k}$, which gives

$$\mathcal{T}_{i,j}\mathcal{T}_{j,k} \subseteq \mathcal{T}_{i,k}. \quad (9)$$

Since $A, E_i^* \in \mathcal{T}$, we have $AE_{i_2}^*AE_{i_3}^* \cdots AE_{i_{p-1}}^*A \in \mathcal{T}$, which follows that

$$A_{i_1, i_2}A_{i_2, i_3} \cdots A_{i_{p-2}, i_{p-1}}A_{i_{p-1}, i_p} \in \mathcal{T}_{i_1, i_p}, \quad (10)$$

where $0 \leq i_1, i_2, \dots, i_p \leq 2m+1$.

Lemma 3.2 *For $2i+2 \leq j \leq 2m+1$ and $0 \leq s \leq i+1$, we have*

$$C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s(n-m) \in \mathcal{T}_{2i+2, j}.$$

Proof. We use induction on s (s decreasing from $i+1$ to 0).

By (10), for $j > 2i+2$ we have $A_{2i+2, 2i+3}A_{2i+3, 2i+4} \cdots A_{j-1, j} \in \mathcal{T}_{2i+2, j}$, which yields that

$$C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{i+1}(n-m) \in \mathcal{T}_{2i+2, j}. \quad (11)$$

When $j = 2i+2$ we pick $I_{\binom{m}{m-i-1}} \otimes I_{\binom{n-m}{i+1}} \in \mathcal{T}_{2i+2, 2i+2}$, which also satisfies (11).

Assume that $C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s(n-m) \in \mathcal{T}_{2i+2, j}$. By (9) and (10) we obtain

$$(C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s(n-m))(A_{j, j+1}A_{j+1, j}) \in \mathcal{T}_{2j+2, j}\mathcal{T}_{j, j} \subseteq \mathcal{T}_{2i+2, j}, \quad (12)$$

$$(C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s(n-m))(A_{j, j-1}A_{j-1, j}) \in \mathcal{T}_{2j+2, j}\mathcal{T}_{j, j} \subseteq \mathcal{T}_{2i+2, j}. \quad (13)$$

When j is even, by Lemma 2.2, Lemma 2.3 (iii) and (v), (12) leads to

$$aC_{m-i-1, m-\frac{j}{2}}^{m-\frac{j}{2}}(m) \otimes C_{i+1, \frac{j}{2}}^s(n-m) + bC_{m-i-1, m-\frac{j}{2}}^{m-\frac{j}{2}}(m) \otimes C_{i+1, \frac{j}{2}}^{s-1}(n-m) \in \mathcal{T}_{2i+2, j},$$

where $a = (n - m - s - \frac{j}{2})(\frac{j}{2} - s + 1)$ and $b = (i - s + 2)(\frac{j}{2} - s + 1)$. Similarly when j is odd, (13) yields that

$$a' C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s(n-m) + b' C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{s-1}(n-m) \in \mathcal{T}_{2i+2, j},$$

where $a' = (n - m - s - \lceil \frac{j}{2} \rceil + 1)(\lceil \frac{j}{2} \rceil - s)$ and $b' = (i - s + 2)(\lceil \frac{j}{2} \rceil - s + 1)$. Since $s \leq i + 1 \leq \lceil \frac{j}{2} \rceil$, $(i - s + 2)(\lceil \frac{j}{2} \rceil - s + 1) \neq 0$. Thus we have $C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{s-1}(n-m) \in \mathcal{T}_{2i+2, j}$.

Hence the desired result follows. \square

Lemma 3.3 *The algebra \mathcal{M} is a subalgebra of \mathcal{T} .*

Proof. During this proof we will omit the symbol (m) from matrices in front of “ \otimes ”, and omit $(n - m)$ from matrices behind “ \otimes ”.

In order to get the desired conclusion, we only need to show that $\mathcal{M}_{i, j} \subseteq \mathcal{T}_{i, j}$ for $0 \leq i, j \leq 2m + 1$. Write $\mathcal{M}_{i, j}^t = \{M^t \mid M \in \mathcal{M}_{i, j}\}$ and $\mathcal{T}_{i, j}^t = \{M^t \mid M \in \mathcal{T}_{i, j}\}$. Since $\mathcal{M}_{j, i} = \mathcal{M}_{i, j}^t$ and $\mathcal{T}_{j, i} = \mathcal{T}_{i, j}^t$, it suffices to prove $\mathcal{M}_{i, j} \subseteq \mathcal{T}_{i, j}$ for $i \leq j$. We use induction on i .

Step 1. Show $\mathcal{M}_{0, j} \subseteq \mathcal{T}_{0, j}$ ($0 \leq j \leq 2m + 1$).

According to (8),

$$\mathcal{M}_{0, j} = \text{Span}\{C_{m, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{0, \lceil \frac{j}{2} \rceil}^0, \quad 0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\}.$$

For any $l \in \{0, 1, \dots, m - \lfloor \frac{j}{2} \rfloor\}$,

$$C_{m, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{0, \lceil \frac{j}{2} \rceil}^0 = \binom{m - \lfloor \frac{j}{2} \rfloor}{l} W_{m-\lfloor \frac{j}{2} \rfloor, m}^t \otimes W_{0, \lceil \frac{j}{2} \rceil},$$

while by (1) and Lemma 2.2,

$$A_{0,1} A_{1,2} \cdots A_{j-1, j} = \lfloor \frac{j}{2} \rfloor! \lceil \frac{j}{2} \rceil! W_{m-\lfloor \frac{j}{2} \rfloor, m}^t \otimes W_{0, \lceil \frac{j}{2} \rceil}.$$

Hence we get $\mathcal{M}_{0, j} \subseteq \mathcal{T}_{0, j}$ from (10).

Step 2. Assume that $\mathcal{M}_{p, j} \subseteq \mathcal{T}_{p, j}$ for $p \leq 2i$. We will show that $\mathcal{M}_{2i+1, j} \subseteq \mathcal{T}_{2i+1, j}$ and $\mathcal{M}_{2i+2, j} \subseteq \mathcal{T}_{2i+2, j}$.

Step 2.1. Show $\mathcal{M}_{2i+1, j} \subseteq \mathcal{T}_{2i+1, j}$ ($2i + 1 \leq j \leq 2m + 1$).

It suffices to prove

$$C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s \in \mathcal{T}_{2i+1, j}, \quad (14)$$

where $0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor$, $0 \leq s \leq i + 1$.

By inductive hypothesis,

$$C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i, \lceil \frac{j}{2} \rceil}^s \in \mathcal{M}_{2i, j} \subseteq \mathcal{T}_{2i, j}, \quad 0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor, \quad 0 \leq s \leq i.$$

Since

$$A_{2i, 2i+1}^t = I_{\binom{m}{m-i}} \otimes W_{i, i+1}^t \in \mathcal{M}_{2i, 2i+1}^t \subseteq \mathcal{T}_{2i, 2i+1}^t,$$

we have

$$(I_{\binom{m}{m-i}} \otimes W_{i, i+1}^t)(C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i, \lceil \frac{j}{2} \rceil}^s) \in \mathcal{T}_{2i, 2i+1}^t \mathcal{T}_{2i, j} \subseteq \mathcal{T}_{2i+1, j},$$

which by Lemma 2.3 (ii) follows that (14) holds for $0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor$, $0 \leq s \leq i$.

By (10), for $j > 2i + 1$ we get

$$A_{2i+1,2i+2} A_{2i+2,2i+3} \cdots A_{j-1,j} \in \mathcal{T}_{2i+1,j},$$

which yields that

$$W_{m-\lfloor \frac{j}{2} \rfloor, m-i}^t \otimes W_{i+1, \lceil \frac{j}{2} \rceil} = C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor} \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{i+1} \in \mathcal{T}_{2i+1,j}. \quad (15)$$

When $j = 2i + 1$ we pick $I_{\binom{m-i}{m-i}} \otimes I_{\binom{n-m}{i+1}} \in \mathcal{T}_{2i+1,2i+1}$, which also satisfies (15).

Case 1. $j = 2m + 1$ or $2m$.

In this case, (15) implies that $C_{m-i,0}^0 \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{i+1} \in \mathcal{T}_{2i+1,j}$, which means (14) holds for $l = 0$ and $s = i + 1$.

Case 2. $j \leq 2m - 1$.

For $j + 1 \leq k \leq 2m$, let

$$N_{j,k} = A_{j,j+1} A_{j+1,j+2} \cdots A_{k-1,k}.$$

Again by (10)

$$A_{2i+1,2i+2} A_{2i+2,2i+3} \cdots A_{j-1,j} N_{j,k} N_{j,k}^t \in \mathcal{T}_{2i+1,j}.$$

By (1), Lemma 2.2 and Lemma 2.3(i), (iii), we obtain

$$c C_{m-i, m-\lfloor \frac{k}{2} \rfloor}^{m-\lfloor \frac{k}{2} \rfloor} \otimes \left(\sum_{h=\max(0, i+1+\lceil \frac{j}{2} \rceil - \lceil \frac{k}{2} \rceil)}^{i+1} \binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - i - 1 - \lceil \frac{j}{2} \rceil + h} C_{i+1, \lceil \frac{j}{2} \rceil}^h \right) \in \mathcal{T}_{2i+1,j}, \quad (16)$$

where $c = (\lfloor \frac{k}{2} \rfloor - i)! (\lceil \frac{k}{2} \rceil - i - 1)! (\lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor)! (\lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil)! \neq 0$. We consider the coefficient of $C_{m-i, m-\lfloor \frac{k}{2} \rfloor}^{m-\lfloor \frac{k}{2} \rfloor} \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^{i+1}$, which is $c \binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil}$. Since $0 \leq 2i + 1 \leq j \leq k - 1 \leq 2m - 1$ and $n \geq 3m$, we get

$$n - m - i - 1 - \lceil \frac{j}{2} \rceil \geq n - m - m - \lceil \frac{j}{2} \rceil \geq m - \lceil \frac{j}{2} \rceil \geq \lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil \geq 0,$$

and so $c \binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil} \neq 0$. Since (14) holds for $s \in \{0, 1, \dots, i\}$, (14) also holds for $s = i + 1$ by (15) and (16).

Step 2.2. Show $\mathcal{M}_{2i+2,j} \subseteq \mathcal{T}_{2i+2,j}$ ($2i + 2 \leq j \leq 2m + 1$).

It suffices to prove

$$C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s \in \mathcal{T}_{2i+2,j}, \quad 0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor, \quad 0 \leq s \leq i + 1. \quad (17)$$

By the inductive assumption, for $0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor$ and $0 \leq s \leq i + 1$,

$$C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s \in \mathcal{M}_{2i+1,j} \subseteq \mathcal{T}_{2i+1,j}.$$

Since

$$A_{2i+1,2i+2}^t = W_{m-i-1, m-i} \otimes I_{\binom{n-m}{i+1}} \in \mathcal{T}_{2i+1,2i+2}^t,$$

by (9) we have

$$\begin{aligned} & (W_{m-i-1, m-i} \otimes I_{\binom{n-m}{i+1}}) (C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s) \\ &= (W_{m-i-1, m-i} C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l) \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s \\ &\in \mathcal{T}_{2i+1,2i+2}^t \mathcal{T}_{2i+1,j} \\ &\subseteq \mathcal{T}_{2i+2,j}. \end{aligned} \quad (18)$$

By Lemma 2.3 (iv),

$$W_{m-i-1, m-i} C_{m-i, m-\lfloor \frac{j}{2} \rfloor}^l = (i+1-l) C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^l + (m - \lfloor \frac{j}{2} \rfloor - l + 1) C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{l-1}.$$

Thus (18) leads to

$$[(i+1-l) C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^l + (m - \lfloor \frac{j}{2} \rfloor - l + 1) C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{l-1}] \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s \in \mathcal{T}_{2i+2, j}, \quad (19)$$

where $0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor$, $0 \leq s \leq i+1$. Since the coefficient of $C_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}^{l-1} \otimes C_{i+1, \lceil \frac{j}{2} \rceil}^s$ in (19) is $m - \lfloor \frac{j}{2} \rfloor - l + 1 \neq 0$, by Lemma 3.2 we get (17).

Hence the desired result follows. \square

Theorem 3.4 *Let $J(n, m, m+1)$ be the incidence graph of Johnson geometry with $n \geq 3m$. Let $\mathcal{T} = \mathcal{T}(x)$ be the Terwilliger algebra of $J(n, m, m+1)$ with respect to an m -subset x and \mathcal{M} be the corresponding algebra defined in (8). Then $\mathcal{T} = \mathcal{M}$.*

Proof. Combining Lemma 3.1 and Lemma 3.3, the proof of Theorem 3.4 is completed. \square

Corollary 3.5 *With reference to Theorem 3.4 $J(n, m, m+1)$ is thin with respect to x .*

Proof. By Theorem 3.4 we get

$$E_i^* \mathcal{T} E_i^* = \text{Span}\{L(C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^l(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^s(n-m)), 0 \leq l \leq m - \lfloor \frac{i}{2} \rfloor, 0 \leq s \leq \lceil \frac{i}{2} \rceil\},$$

where $i = 0, 1, \dots, D(x)$. Since each element of $E_i^* \mathcal{T} E_i^*$ is symmetric, we get the conclusion from [15, Theorem 13]. \square

4 The basis of the Terwilliger algebra

In this section we shall determine the basis and the dimension of \mathcal{T} .

Theorem 4.1 *Let $G_{i,j} = \{g \mid H_{m-\lfloor \frac{j}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \neq 0\}$, $R_{i,j} = \{r \mid H_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^r(n-m) \neq 0\}$, and \mathcal{T} be as in Theorem 3.4. Then we have*

$$\{L(H_{m-\lfloor \frac{j}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \otimes H_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^r(n-m)), g \in G_{i,j}, r \in R_{i,j}\}_{i,j=0}^{2m+1} \quad (20)$$

as well as

$$\{L(C_{m-\lfloor \frac{j}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^l(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^s(n-m)), l \in G_{i,j}, s \in R_{i,j}\}_{i,j=0}^{2m+1} \quad (21)$$

are two bases of \mathcal{T} .

Proof. Without loss of generality, we assume $i \leq j$. Since $H_{i,j}^l(v) \neq 0$ if and only if $\max(0, i+j-v) \leq l \leq \min(i, j)$, we have $\lceil \frac{i}{2} \rceil - |R_{i,j}| + 1 \leq r \leq \lceil \frac{i}{2} \rceil$ when $r \in R_{i,j}$. By (5) we obtain

$$C_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^r(n-m) = \sum_{h=r}^{\lceil \frac{i}{2} \rceil} \binom{h}{r} H_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}^h(n-m), \quad (22)$$

which implies that $H_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^r(n-m)$ ($r \in R_{i,j}$) is a linear combination of $\{C_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^s(n-m)\}_{s \in R_{i,j}}$. Similarly, $H_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m)$ ($g \in G_{i,j}$) can be expressed as a linear combination of $\{C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^l(m)\}_{l \in G_{i,j}}$. Hence

$$H_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \otimes H_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^r(n-m) \in \mathcal{M}_{i,j}.$$

Again by (5), for $0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor$ and $0 \leq s \leq \lceil \frac{i}{2} \rceil$,

$$\begin{aligned} & C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^l(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^s(n-m) \\ &= \sum_{g=l}^{m-\lfloor \frac{j}{2} \rfloor} \binom{g}{l} H_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \otimes \sum_{r=s}^{\lceil \frac{i}{2} \rceil} \binom{r}{s} H_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^r(n-m). \end{aligned}$$

Observe that $H_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \otimes H_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^r(n-m)$ ($g \in G_{i,j}, r \in R_{i,j}$) are linearly independent. Then $\{H_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^g(m) \otimes H_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^r(n-m)\}_{g \in G_{i,j}, r \in R_{i,j}}$ is a basis of $\mathcal{M}_{i,j}$. Therefore (20) is a basis of \mathcal{T} .

Furthermore by (22) we can get $\{C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}^l(m) \otimes C_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}^s(n-m)\}_{l \in G_{i,j}, s \in R_{i,j}}$ is also a basis of $\mathcal{M}_{i,j}$, which follows that (21) is a basis of \mathcal{T} .

This ends our proof. \square

Corollary 4.2 *With reference to Theorem 3.4 we get the dimension of \mathcal{T} is*

$$\dim \mathcal{T} = \begin{cases} \frac{1}{12}(m+1)(m+2)(m+3)(3m+10) - 4, & \text{if } n = 3m, \\ \frac{1}{12}(m+1)(m+2)(m+3)(3m+10) - 1, & \text{if } n = 3m + 1, \\ \frac{1}{12}(m+1)(m+2)(m+3)(3m+10), & \text{if } n \geq 3m + 2. \end{cases}$$

Proof. By Theorem 4.1 we get

$$\begin{aligned} \dim \mathcal{T} &= \sum_{i,j=0}^{2m+1} |G_{i,j}| |R_{i,j}| \\ &= \sum_{i,j=0}^{2m+1} (\min(m - \lfloor \frac{i}{2} \rfloor, m - \lfloor \frac{j}{2} \rfloor) - \max(0, m - \lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor) + 1) \\ &\quad \cdot (\min(\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil) - \max(0, \lceil \frac{i}{2} \rceil + \lceil \frac{j}{2} \rceil - n + m) + 1). \end{aligned}$$

By zigzag calculation, we get the desired result. \square

5 Concluding Remark

We conclude this paper with the following remarks:

(i) Let $J(n, m)$ be the Johnson graph with $n \geq 3m$. Fix a vertex x of $J(n, m)$. Let $\mathcal{T}' = \mathcal{T}'(x)$ and $\mathcal{T} = \mathcal{T}(x)$ be the Terwilliger algebra of $J(n, m)$ and $J(n, m, m+1)$ with respect to x , respectively. Since $\bigoplus_{i,j=0}^m E_{2i}^*(x) \mathcal{T} E_{2j}^*(x)$ is an algebra, $\{L(H_{m-i, m-j}^g(m) \otimes H_{i,j}^r(n-m)), g \in G_{2i, 2j}, r \in R_{2i, 2j}\}_{i,j=0}^m$ is a basis of $\bigoplus_{i,j=0}^m E_{2i}^*(x) \mathcal{T} E_{2j}^*(x)$ by Theorem 4.1. By [10, Definition 4.2, Lemma 4.4, Theorem 5.9] this basis coincides with that of \mathcal{T}' , which implies that $\mathcal{T}' \simeq \bigoplus_{i,j=0}^m E_{2i}^*(x) \mathcal{T} E_{2j}^*(x)$.

(ii) Using the same method, the Terwilliger algebra of $J(n, m, m+1)$ with respect to an $(m+1)$ -subset may be determined.

Acknowledgement

The authors would like to thank Professor Hiroshi Suzuki for proposing this problem and for his many helpful suggestions. This research is partially supported by NSF of China (10871027), NCET-08-0052, and the Fundamental Research Funds for the Central Universities of China.

References

- [1] P. Balmaceda, M. Oura, The Terwilliger algebras of the group association schemes of S_5 and A_5 , Kyushu J. Math. 48 (1994) 221–231.
- [2] E. Bannai, A. Munemasa, The Terwilliger algebras of group association schemes, Kyushu J. Math. 49 (1995) 93–102.
- [3] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, Heidelberg, 1989.
- [4] J. S. Caughman IV, The Terwilliger algebra for bipartite P - and Q -polynomial association schemes, Discrete Math. 196 (1999) 65–95.
- [5] J. S. Caughman, M. S. MacLean, P. Terwilliger, The Terwilliger algebra of an almost-bipartite P - and Q -polynomial association scheme, Discrete Math. 292 (2005) 17–44.
- [6] B. Curtin, The Terwilliger algebra of a 2-homogeneous bipartite distance-regular graph, J. Combin. Theory Ser. A 81 (2001) 125–141.
- [7] J. Go, The Terwilliger algebra of the hypercube, Europ. J. Combin. 23 (2002) 399–429.
- [8] A. Hiraki, A characterization of the doubled Grassmann graphs, the doubled Odd graphs, and the Odd graphs by strongly closed subgraphs, Europ. J. of Combin. 24 (2003) 161–171.
- [9] F. Levstein, C. Maldonado, D. Penazzi, The Terwilliger algebra of a Hamming scheme $H(d, q)$, Europ. J. Combin. 27 (2006) 1–10.
- [10] F. Levstein, C. Maldonado, The Terwilliger algebra of the Johnson schemes, Discrete Mathematics 307 (2007) 1621–1635.
- [11] M. Mohammad-Noori, N. Ghareghani, E. Ghorbani, Intersection matrices and the Johnson scheme, arXiv:0902.4367v3[math.CO].
- [12] P. Terwilliger, The subconstituent algebra of an association scheme I, J. Algebr. Comb. 1 (1992) 363–388.
- [13] P. Terwilliger, The subconstituent algebra of an association scheme II, J. Algebr. Comb. 2 (1993) 73–103.
- [14] P. Terwilliger, The subconstituent algebra of an association scheme III, J. Algebr. Comb. 2 (1993) 177–210.
- [15] P. Terwilliger, Algebraic Graph Theory, Hand-Written Lecture Note of Paul Terwilliger, Rewritten and Added Comments by H. Suzuki.
<http://subsite.icu.ac.jp/people/hsuzuki/lecturenote/>
- [16] M. Tomiyama and N. Yamazaki, The subconstituent algebra of a strongly regular graph, Kyushu J. Math. 48 (1994) 323–334.